

# $n$ -particle quantum statistics on graphs

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QMath13 – 10/16

# Outline

- 1 Quantum statistics
- 2 Statistics on graphs
- 3 3-connected graphs

# Quantum statistics

Single particle space configuration space  $X$ .

## Two particle statistics - alternative approaches:

- Quantize  $X^{\times 2}$  and restrict Hilbert space to the symmetric or anti-symmetric subspace.

$$\psi(x_1, x_2) = \pm \psi(x_2, x_1) \quad (1)$$

Bose-Einstein/Fermi-Dirac statistics.

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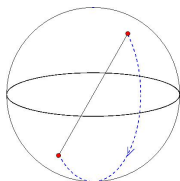
$$\psi(x_1, x_2) = \pm \psi(x_2, x_1) \quad (1)$$

Bose-Einstein/Fermi-Dirac statistics.

- (Leinaas and Myrheim '77)  
Treat particles as indistinguishable,  $\psi(x_1, x_2) \equiv \psi(x_2, x_1)$ .  
Quantize two particle configuration space.

## Bose-Einstein and Fermi-Dirac statistics

Two indistinguishable particles in  $\mathbb{R}^3$ . At constant separation relative coordinate lies on projective plane.



Exchanging particles corresponds to rotating relative coordinate around closed loop  $p$ .

$p$  is not contractible but  $p^2$  is contractible.

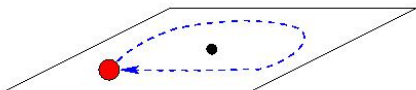
To associate a phase factor  $e^{i\theta}$  to  $p$  requires  $(e^{i\theta})^2 = 1$ .

Quantizing configuration space with phase  $\pi$  corresponds to *Fermi-Dirac statistics* and phase 0 to *Bose-Einstein statistics*.

# Anyon statistics

Pair of indistinguishable particles in  $\mathbb{R}^2$ .

- Particles not coincident.
- Relative position coordinate in  $\mathbb{R}^2 \setminus \mathbf{0}$ .
- Exchange paths are closed loops about  $\mathbf{0}$  in relative coordinate.
- *Any* phase factor  $e^{i\theta}$  can be associated with a primitive exchange path.



## Definition

Configuration space of  $n$  indistinguishable particles in  $X$ ,

$$C_n(X) = (X^{\times n} - \Delta_n) / S_n$$

where  $\Delta_n = \{x_1, \dots, x_n \mid x_i = x_j \text{ for some } i \neq j\}$ .

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1st homology groups of  $C_n(\mathbb{R}^d)$ :

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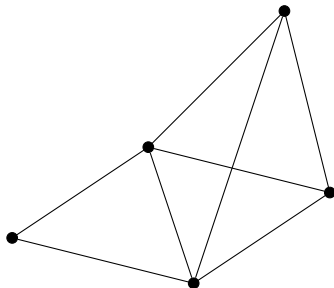
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Any single phase  $\theta$  can be associated to primitive exchange paths – **anyon** statistics.
- $H_1(C_n(\mathbb{R})) = 1$   
particles cannot be exchanged.

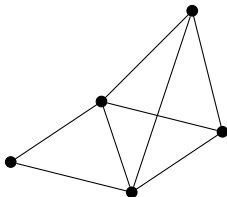
What happens on a graph where the underlying space has arbitrarily complex topology?



# Graph connectivity

- Given a connected graph  $\Gamma$  a  *$k$ -cut* is a set of  $k$  vertices whose removal makes  $\Gamma$  disconnected.
- $\Gamma$  is  *$k$ -connected* if the minimal cut is size  $k$ .
- Theorem** (Menger) For a  $k$ -connected graph there exist at least  $k$  independent paths between every pair of vertices.

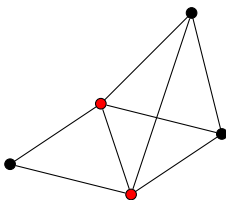
Example:



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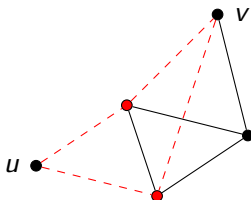


Two cut

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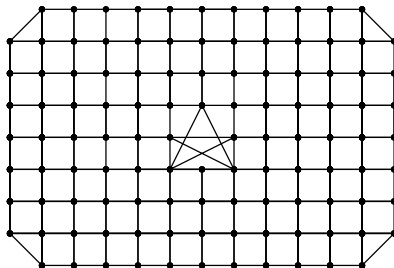
Two independent paths joining  $u$  and  $v$ .

# Features of graph statistics

**3-connected graphs:** statistics only depend on whether the graph is **planar** (Anyons) or **non-planar** (Bosons/Fermions).

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A planar lattice with a small section that is non-planar is locally planar but has Bose/Fermi statistics.

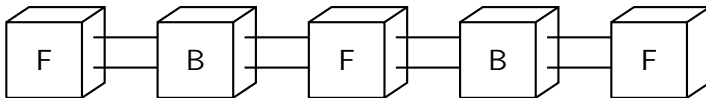


# Features of graph statistics

**2-connected graphs:** statistics complex but independent of the number of particles.

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For example, one could construct a chain of 3-connected non-planar components where particles behave with alternating Bose/Fermi statistics.

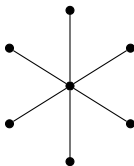
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Example, star with  $E$  edges.

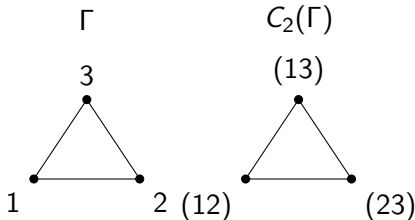


no. of anyon phases

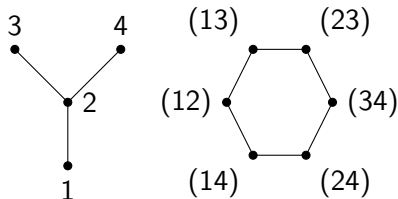
$$\binom{n + E - 2}{E - 1} (E - 2) - \binom{n + E - 2}{E - 2} + 1 .$$

# Basic cases

For 2 particles.



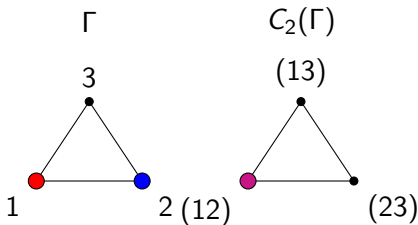
Exchange of 2 particles around loop  $c$ ; one free phase  $\phi_{c2}$ .



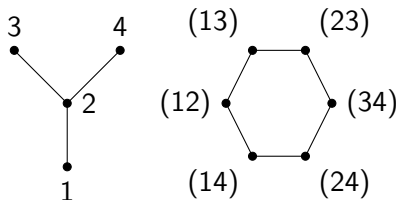
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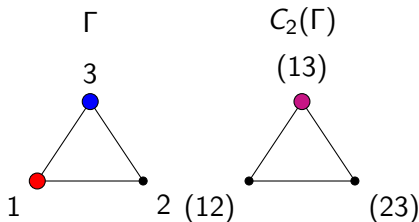
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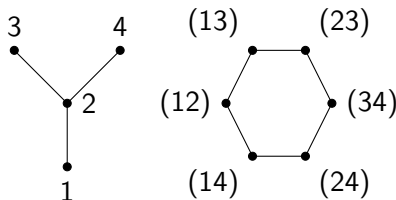
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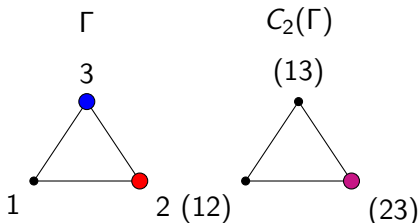
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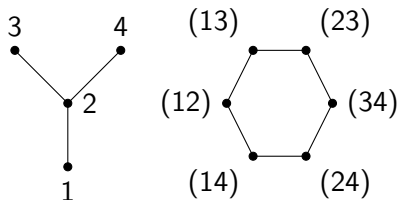
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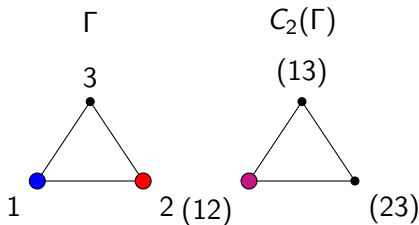


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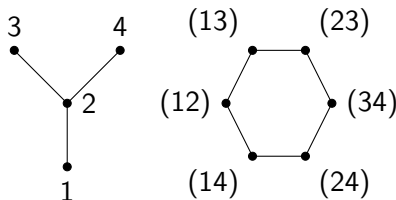


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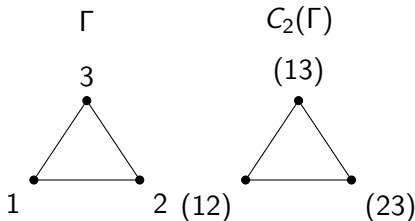
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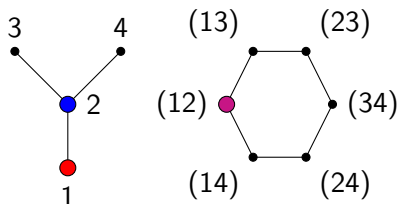
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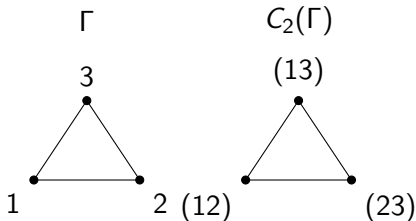
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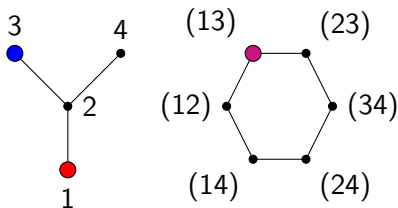
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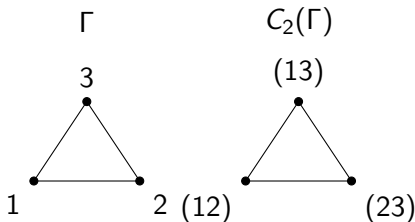
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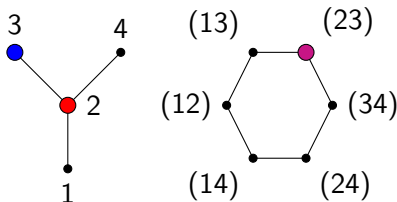
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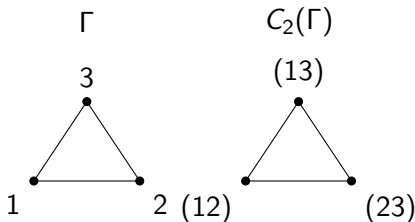
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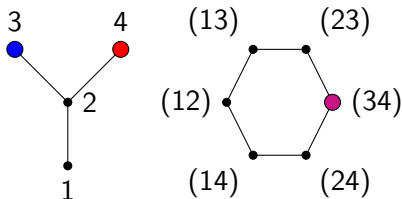
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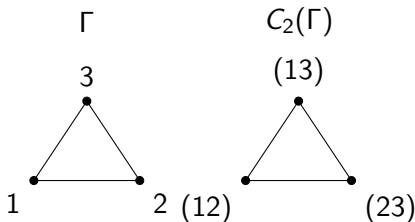
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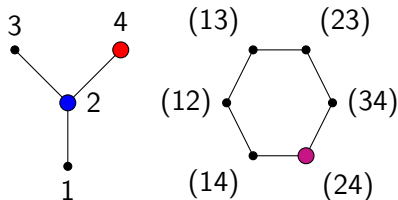
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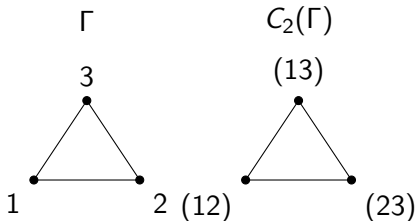
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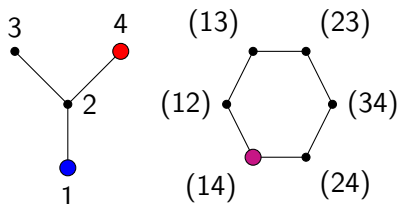
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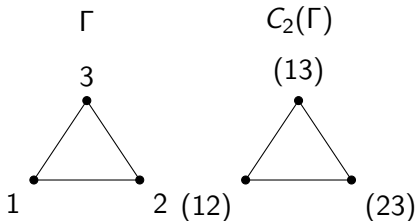
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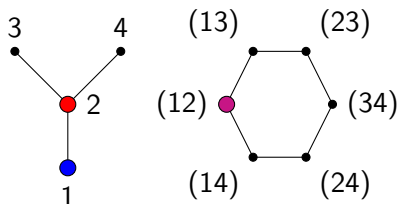
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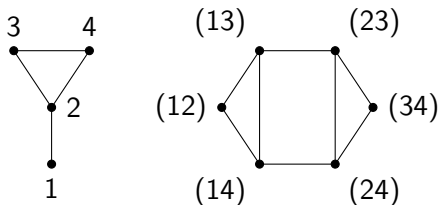
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# Lasso graph

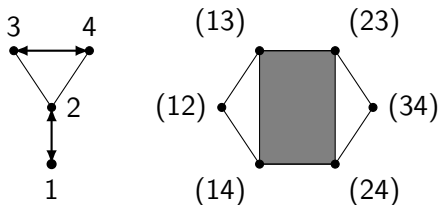


Identify three 2-particle cycles:

- (i) Rotate both particles around loop  $c$ ; phase  $\phi_{c,2}$ .
- (ii) Exchange particles on Y-subgraph; phase  $\phi_Y$ .
- (iii) Rotate one particle around loop  $c$  other particle at vertex 1;  
 $(1,2) \rightarrow (1,3) \rightarrow (1,4) \rightarrow (1,2)$ , phase  $\phi_{c,1}^1$ .

Relation from contactable 2-cell  $\phi_{c,2} = \phi_{c,1}^1 + \phi_Y$ .

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Let  $c$  be a loop. What is the relation between  $\phi_{c,1}^u$  and  $\phi_{c,1}^v$ ?

(a)  $u$  and  $v$  joined by path disjoint with  $c$ .

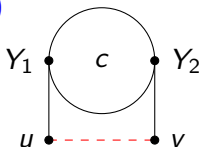
$\phi_{c,1}^u = \phi_{c,1}^v$  as exchange cycles homotopy equivalent.

(b)  $u$  and  $v$  *only* joined by paths through  $c$ .

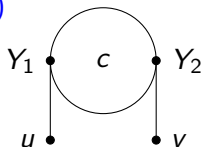
Two lasso graphs so  $\phi_{c,2} = \phi_{c,1}^u + \phi_{Y_1}$  &  $\phi_{c,2} = \phi_{c,1}^v + \phi_{Y_2}$ .

Hence  $\phi_{c,1}^u - \phi_{c,1}^v = \phi_{Y_2} - \phi_{Y_1}$ .

(a)



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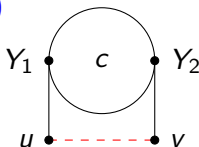
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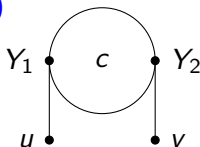
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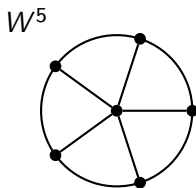
(b)



- Relations between phases involving  $c$  encoded in phases  $\phi_Y$ .  
 $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$ , where  $A$  determined by  $Y$ -cycles.
- In (a) we have a  $\mathcal{B}$  subgraph & using (b) also  $\phi_{Y_1} = \phi_{Y_2}$ .

## 3-connected graphs

The prototypical 3-connected graph is a *wheel*  $W^k$ .



### Theorem (Wheel theorem)

*Let  $\Gamma$  be a simple 3-connected graph different from a wheel. Then for some edge  $e \in \Gamma$  either  $\Gamma \setminus e$  or  $\Gamma / e$  is simple and 3-connected.*

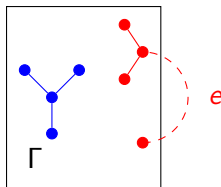
- $\Gamma \setminus e$  is  $\Gamma$  with the edge  $e$  removed.
- $\Gamma / e$  is  $\Gamma$  with  $e$  contracted to identify its vertices.

## Lemma

*For 3-connected simple graphs all phases  $\phi_\gamma$  are equal up to a sign.*

**Sketch proof.** The lemma holds on  $K_4$  (minimal wheel). By wheel theorem we need to show that adding an edge or expanding a vertex any new phases  $\phi_\gamma$  are the same as an original phase.

*Adding an edge:*  $\Gamma \cup e$

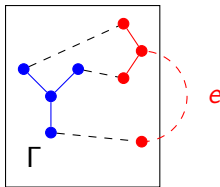


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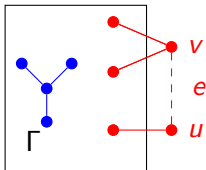
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**Vertex expansion:** Split vertex of degree  $> 3$  into two vertices  $u$  and  $v$  joined by a new edge  $e$ .



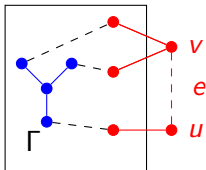


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## Theorem

*For a 3-connected simple graph,  $H_1(C_2(\Gamma)) = \mathbb{Z}^{\beta_1(\Gamma)} \oplus A$ , where  $A = \mathbb{Z}_2$  for non-planar graphs and  $A = \mathbb{Z}$  for planar graphs.*

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## Proof.

- For  $K_5$  and  $K_{3,3}$  every phase  $\phi_Y = 0$  or  $\pi$ . By Kuratowski's theorem a non-planar graph contains a subgraph which is isomorphic to  $K_5$  or  $K_{3,3}$ .

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- For planar graphs the anyon phase can be introduced by drawing the graph in the plane and integrating the anyon vector potential  $\frac{\alpha}{2\pi} \hat{z} \times \frac{r_1 - r_2}{|r_1 - r_2|^2}$  along the edges of the two-particle graph.

# Classification of graph statistics

*Ko & Park (2011)*

$$H_1(C_n(G)) = \mathbb{Z}^{N_1(G)+N_2(G)+N_3(G)+\beta_1(G)} \oplus \mathbb{Z}_2^{N'_3(G)} \quad (2)$$

- $N_1(G)$  sum over one cuts  $j$  of  $N(n, G, j)$ .

$$N(n, G, j) = \binom{n + \mu_j - 2}{n - 1} (\mu(j) - 2) - \binom{n + \mu_j - 2}{n} - (v_j - \mu_j - 1)$$

$\mu_j$  # components of  $G \setminus j$ .

- $N_2(G)$  sum over two connected components of  $G$ .
- $N_3(G)$  # 3-connected planar components of  $G$ .
- $N'_3(G)$  # 3-connected non-planar components of  $G$ .
- $\beta_1(G)$  # of loops of  $G$ .

# Summary

- Classification of abelian quantum statistics on graphs by graph theoretic argument.
- Physical insight into dependence of statistics on graph connectivity.
- Interesting new features of graph statistics.
- Are there phenomena associated with new forms of graph statistics - e.g. fractional quantum Hall experiment on network?



JH, JP Keating, JM Robbins and A Sawicki, “ $n$ -particle quantum statistics on graphs,” *Commun. Math. Phys.* (2014) **330** 1293–1326 [arXiv:1304.5781](#)



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